Problem 3.1

(a) Show that the set of all square-integrable functions is a vector space (refer to Section A.1 for the definition). *Hint:* The main point is to show that the sum of two square-integrable functions is itself square-integrable. Use Equation 3.7. Is the set of all *normalized* functions a vector space?

(b) Show that the integral in Equation 3.6 satisfies the conditions for an inner product (Section A.2).

Solution

In order for a collection of vectors $\mathcal{V}$ to be a vector space over the complex numbers $\mathbb{C}$, the vector addition and scalar multiplication operations defined on it must satisfy the following ten properties.

(A1) $\mathbf{x}+\mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A2) $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$.

(A3) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A4) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x}+\mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

(A5) For each $\mathbf{x} \in \mathcal{V}$, there is an element $(-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$.

(M1) $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathbb{C}$ and $\mathbf{x} \in \mathcal{V}$.

(M2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in \mathbb{C}$ and every $\mathbf{x} \in \mathcal{V}$.

(M3) $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$ for every $\alpha \in \mathbb{C}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(M4) $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$ for all $\alpha, \beta \in \mathbb{C}$ and every $\mathbf{x} \in \mathcal{V}$.

(M5) $1\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

Part (a)

Let $\mathcal{V}$ be the set of all square-integrable functions. Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions in $\mathcal{V}$ and that α and β are complex scalars.

$$\int_a^b |f(x)|^2\,dx = \langle f\,|\,f\rangle = C_1 < \infty$$

$$\int_a^b |g(x)|^2\,dx = \langle g\,|\,g\rangle = C_2 < \infty$$

$$\int_a^b |h(x)|^2\,dx = \langle h\,|\,h\rangle = C_3 < \infty$$

C_1, C_2, and C_3 are real nonnegative constants.

Property A1

Check to see if the sum of $f(x)$ and $g(x)$ is also in $\mathcal{V}$.

$$\begin{aligned}
\int_a^b |f(x)+g(x)|^2\,dx &= \int_a^b [f(x)+g(x)]^*[f(x)+g(x)]\,dx \\
&= \int_a^b [f^*(x)+g^*(x)][f(x)+g(x)]\,dx \\
&= \int_a^b [f^*(x)f(x)+f^*(x)g(x)+g^*(x)f(x)+g^*(x)g(x)]\,dx \\
&= \int_a^b f^*(x)f(x)\,dx + \int_a^b f^*(x)g(x)\,dx + \int_a^b g^*(x)f(x)\,dx + \int_a^b g^*(x)g(x)\,dx \\
&= \langle f\,|\,f\rangle + \langle f\,|\,g\rangle + \langle g\,|\,f\rangle + \langle g\,|\,g\rangle \\
&= \langle f\,|\,f\rangle + \langle f\,|\,g\rangle + \langle f\,|\,g\rangle^* + \langle g\,|\,g\rangle \\
&= \langle f\,|\,f\rangle + 2\,\mathrm{Re}\,\langle f\,|\,g\rangle + \langle g\,|\,g\rangle
\end{aligned}$$

Consider the modulus of both sides.

$$\left|\int_a^b |f(x)+g(x)|^2\,dx\right| = |\langle f\,|\,f\rangle + 2\,\mathrm{Re}\,\langle f\,|\,g\rangle + \langle g\,|\,g\rangle|$$

Use the triangle inequality.

$$\begin{aligned}
\left|\int_a^b |f(x)+g(x)|^2\,dx\right| &\le |\langle f\,|\,f\rangle| + |2\,\mathrm{Re}\,\langle f\,|\,g\rangle| + |\langle g\,|\,g\rangle| \\
&= |\langle f\,|\,f\rangle| + 2\,|\mathrm{Re}\,\langle f\,|\,g\rangle| + |\langle g\,|\,g\rangle| \\
&\le |\langle f\,|\,f\rangle| + 2\,|\langle f\,|\,g\rangle| + |\langle g\,|\,g\rangle| \\
&= |\langle f\,|\,f\rangle| + 2\left|\int_a^b f^*(x)g(x)\,dx\right| + |\langle g\,|\,g\rangle|
\end{aligned}$$

Use the Schwarz inequality (Equation 3.7 on page 93 or Equation A.27 on page 467).

$$\begin{aligned}
\left|\int_a^b |f(x)+g(x)|^2\,dx\right| &\le |\langle f\,|\,f\rangle| + 2\sqrt{\int_a^b |f(x)|^2\,dx \int_a^b |g(x)|^2\,dx} + |\langle g\,|\,g\rangle| \\
&= |\langle f\,|\,f\rangle| + 2\sqrt{\langle f\,|\,f\rangle\langle g\,|\,g\rangle} + |\langle g\,|\,g\rangle| \\
&= |C_1| + 2\sqrt{(C_1)(C_2)} + |C_2| \\
&= C_1 + 2\sqrt{C_1C_2} + C_2 \\
&< \infty
\end{aligned}$$

Property A1 is satisfied.

Property A2

The associative law of addition holds for complex numbers, so

$$[f(x) + g(x)] + h(x) = f(x) + [g(x) + h(x)].$$

Property A2 is satisfied.

Property A3

The commutative law of addition holds for complex numbers, so

$$f(x) + g(x) = g(x) + f(x).$$

Property A3 is satisfied.

Property A4

There is in fact a zero function in $\mathcal{V}$ because

$$\int_a^b |0|^2\, dx = \langle 0 \,|\, 0 \rangle = 0 < \infty.$$

Adding 0 to any function in $\mathcal{V}$, say $f(x)$, results in that same function: $f(x) + 0 = f(x)$. Property A4 is satisfied.

Property A5

For any function in $\mathcal{V}$, say $f(x)$, there exists an additive inverse, $-f(x)$, because

$$\int_a^b |-f(x)|^2\, dx = \int_a^b |f(x)|^2\, dx = \langle f \mid f \rangle = C_1 < \infty.$$

Adding any function with its additive inverse yields the zero function: $f(x) + [-f(x)] = 0$. Property A5 is satisfied.

Property M1

$\alpha f(x)$ is also a function in $\mathcal{V}$ because

$$\int_a^b |\alpha f(x)|^2\, dx = \int_a^b |\alpha|^2 |f(x)|^2\, dx = |\alpha|^2 \int_a^b |f(x)|^2\, dx = |\alpha|^2 \langle f \mid f \rangle = |\alpha|^2 C_1 < \infty.$$

Property M1 is satisfied.

Property M2

The associative law of multiplication holds for complex numbers, so property M2 is satisfied.

$$(\alpha\beta) f(x) = \alpha[\beta f(x)].$$

Property M3

The distributive law holds for complex numbers, so

$$\alpha[f(x) + g(x)] = \alpha f(x) + \alpha g(x).$$

Property M3 is satisfied.

Property M4

The distributive law holds for complex numbers, so

$$(\alpha + \beta)f(x) = \alpha f(x) + \beta f(x).$$

Property M4 is satisfied.

Property M5

1 is known as the multiplicative identity because

$$1f(x) = 1 \cdot f(x) = f(x).$$

Property M5 is satisfied. All ten properties are satisfied, so the set of all square-integrable functions is a vector space over the complex numbers. The set of all normalized functions is not a vector space because it does not include the zero function:

$$\int_a^b |0|^2 \, dx = \langle 0 \,|\, 0 \rangle = 0 \neq 1.$$

Part (b)

Equation A.19, Equation A.20, and Equation A.21 on page 467 are the conditions that an inner product has to satisfy.

$$\langle \beta \,|\, \alpha \rangle = \langle \alpha \,|\, \beta \rangle^* \tag{A.19}$$

$$\langle \alpha \,|\, \alpha \rangle \geq 0, \quad \text{and } \langle \alpha \,|\, \alpha \rangle = 0 \Leftrightarrow |\alpha\rangle = |0\rangle \tag{A.20}$$

$$\langle \alpha | \left(b\,|\beta\rangle + c\,|\gamma\rangle \right) = b\,\langle \alpha \,|\, \beta \rangle + c\,\langle \alpha \,|\, \gamma \rangle \tag{A.21}$$

The integral under consideration here is in Equation 3.6.

$$\langle f \,|\, g \rangle = \int_a^b f^*(x) g(x) \, dx \tag{3.6}$$

Note that $f^*(x)g(x)$ is a complex-valued function and can be written as $u(x) + iv(x)$, where $u(x)$ and $v(x)$ are real-valued functions, since x is real.

Start by showing that Equation A.19 is true.

$$\begin{aligned}
\langle f \,|\, g\rangle^* &= \left[\int_a^b f^*(x)g(x)\,dx\right]^* \\
&= \left\{\int_a^b [u(x) + iv(x)]\,dx\right\}^* \\
&= \left[\int_a^b u(x)\,dx + i\int_a^b v(x)\,dx\right]^* \\
&= \int_a^b u(x)\,dx - i\int_a^b v(x)\,dx \\
&= \int_a^b [u(x) - iv(x)]\,dx \\
&= \int_a^b [u(x) + iv(x)]^*\,dx \\
&= \int_a^b [f^*(x)g(x)]^*\,dx \\
&= \int_a^b f(x)g^*(x)\,dx \\
&= \int_a^b g^*(x)f(x)\,dx \\
&= \langle g \,|\, f\rangle
\end{aligned}$$

Now show that Equation A.20 is true. $f(x)$ is a complex-valued function, so it can be written as $U(x) + iV(x)$, where $U(x)$ and $V(x)$ are real-valued functions, since x is real.

$$\begin{aligned}
\langle f \,|\, f\rangle &= \int_a^b f^*(x)f(x)\,dx \\
&= \int_a^b [U(x) + iV(x)]^*[U(x) + iV(x)]\,dx \\
&= \int_a^b [U(x) - iV(x)][U(x) + iV(x)]\,dx \\
&= \int_a^b \{[U(x)]^2 + [V(x)]^2\}\,dx \qquad (1)
\end{aligned}$$

Since $U(x)$ and $V(x)$ are real, the integrand is nonnegative, meaning the integral is nonnegative too.

$$\langle f \,|\, f\rangle \geq 0$$

If $f(x) = 0$, then

$$\langle f \,|\, f\rangle = \int_a^b (0)^*(0)\,dx = 0.$$

The aim now is to show that if

$$\langle f \mid f \rangle = \int_a^b f^*(x) f(x)\, dx = \int_a^b |f(x)|^2\, dx = 0,$$

then $|f(x)|^2 = 0$. Let $F(x) = |f(x)|^2$ and assume that

$$\int_a^b F(x)\, dx = 0 \tag{2}$$

is true. As indicated by equation (1), $F(x)$ is a real and nonnegative function. Assume further that $F(x)$ is a continuous function on $a \leq x \leq b$. By the extreme value theorem, then, $F(x)$ attains an absolute maximum somewhere in this interval, say at $x = c$. Suppose this maximum is positive:

$$F(c) > 0, \quad a \leq c \leq b.$$

Because of the continuity, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - c| < \delta \quad \text{implies} \quad |F(x) - F(c)| < \varepsilon.$$

For $\varepsilon = \frac{1}{2}F(c)$ in particular, this result becomes

$$\begin{aligned} |x - c| < \delta \quad &\text{implies} \quad |F(x) - F(c)| < \frac{1}{2}F(c) \\ -\delta < x - c < \delta \quad &\text{implies} \quad F(c) - F(x) < \frac{1}{2}F(c) \\ c - \delta < x < c + \delta \quad &\text{implies} \quad F(x) > \frac{1}{2}F(c). \end{aligned}$$

Now consider the integral of $F(x)$ from a to b.

$$\int_a^b F(x)\, dx \geq \int_{c-\delta}^{c+\delta} F(x)\, dx > \int_{c-\delta}^{c+\delta} \frac{1}{2}F(c)\, dx = \frac{1}{2}F(c) \int_{c-\delta}^{c+\delta} dx = \frac{1}{2}F(c)(2\delta) = F(c)\delta > 0$$

This contradicts equation (2), so the assumption that $F(c)$ is positive is false; it must be zero then. If the maximum of a continuous nonnegative function is zero, this function is zero for all x: $F(x) = 0$ for $a \leq x \leq b$.

$$|f(x)|^2 = 0$$

Solving this equation for $f(x)$ yields

$$f(x) = 0.$$

There's a problem with this argument if a or b are infinite because the extreme value theorem doesn't guarantee a maximum for $F(x)$ on an open interval, such as $-\infty < x < \infty$. By assuming even further that $F(x) \to 0$ as $x \to \pm\infty$, though, this interval can be made finite and closed ($-d \leq x \leq d$, where d is large enough that $F(x)$ decreases monotonically to zero outside this interval), and the argument above applies. Therefore,

$$\langle f \mid f \rangle = 0 \quad \Leftrightarrow \quad f(x) = 0.$$

Finally, show that Equation A.21 is true. Let α and β be complex constants.

$$\begin{aligned}
\langle f|\,(\alpha\,|g\rangle + \beta\,|h\rangle) &= \int_a^b f^*(x)[\alpha g(x) + \beta h(x)]\,dx \\
&= \int_a^b [\alpha f^*(x) g(x) + \beta f^*(x) h(x)]\,dx \\
&= \int_a^b \alpha f^*(x) g(x)\,dx + \int_a^b \beta f^*(x) h(x)\,dx \\
&= \alpha \int_a^b f^*(x) g(x)\,dx + \beta \int_a^b f^*(x) h(x)\,dx \\
&= \alpha\,\langle f\,|\,g\rangle + \beta\,\langle f\,|\,h\rangle
\end{aligned}$$